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SQIF Array Dynamics

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An analytic perturbation solution is developed for a single cell SQUID and for multi-cell rows with random cell sizes (constrained by having a mirror symmetry about the center junction). The response of a single SQUID to an external RF signal is determined in terms of correlation functions between the phase of the inline effective junction and the circulating current. This yields an RF phase shift that can be calculated as a function of the circuit parameters and the signal frequency. Summing the phase shifted RF voltages of many random cell size SQUIDs in series yields the SQIF RF response.

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SQIF Array Dynamics

1. Abstract

We develop an analytic perturbation solution for single-cell SQUIDs and for multi-cell rows with random cell sizes (constrained by having a mirror symmetry about the center junction). The response of a single SQUID to an external RF signal is determined in terms of correlation functions between the phase of the inline effective junction and the circulating current. This yields an RF phase shift that can be calculated as a function of the circuit parameters and the signal frequency. Summing the phase-shifted RF voltages of many random-cell-size SQUIDs in series yields the SQIF RF response.

2. Characterization of SQIF's as RF Detectors

SQIF arrays were originally developed to detect extremely small static and quasistatic magnetic fields. The paper by Schopohl et. al.[1] showed how it was possible to approximate the RF response of a SQIF array to a low frequency RF signal by using the transfer function (DC voltage vs external magnetic field) in the narrow region near zero magnetic field, where the transfer function is essentially linear. In a SQIF array the interference of the voltages of the random-area-SQUID loops in the array leads to the steep slope of the transfer function near the origin, which should be the only place where the voltages interfere constructively.

To characterize the merit of a SQIF array as an RF detector, it is necessary to take into account both the Johnson-Nyquist noise of the array junctions and the appearance of intrinsic nonlinear effects which determine the range of signal amplitudes in which the transfer function can be considered linear. The widely accepted figure of merit for RF detectors is the 'spur free dynamic range' (SFDR), which is a measure of the range of detector linearity, and is given by the range (in dB) where the detector response (v^2) for two signals that are close in frequency is above the Johnson-Nyquist noise threshold, while the slightly frequency-shifted response due to cubic mixing (at $\omega=2\omega_1-\omega_2$ or $\omega=2\omega_2-\omega_1$) remains below the noise threshold.

The approach that we have taken is to fully determine the RF response of each row of a SQIF array to the magnetic component of an RF field, and sum the voltages to obtain the total RF response. This means that both the amplitude and RF *phase* of the response of each row at the signal frequency are determined, in contrast to the approach in [1] where the phase shifts of each row are taken to be the phase shifts at DC (i.e. those due to the static external flux) which are then assumed to be applicable to low frequency signals. The approach in [1] essentially treats the RF signal as an adiabatically changing flux bias.

To determine the SFDR of an array we start with the response of a single row, and with the simplest case of a single SQUID or a single row SQIF with zero inductance that can be treated as an equivalent effective single SQUID[1]. The linear response of the effective SQUID when the input is a pair of small amplitude signals (amplitudes $\{a_{rf,1}, a_{rf,2}\}$ at frequencies $\{\omega_{rf,1}, \omega_{rf,2}\}$) can be solved, with solutions in the form of a sum of sinusoids at the signal frequencies $\omega_{rf,i}$ with amplitudes A_i and phase shifts $\Psi_{rf,i}$. For high T_c junctions $\beta_C \approx 0$, and this allows good approximate solutions to be obtained for the unperturbed SQUID voltage $\dot{\zeta}_{DC}$ and induced flux $\eta_{DC} = \beta_L i_{circ}$, where the DC subscript refers to the response to ϕ_{DC}, the DC component of the magnetic flux threading the SQUID loop (the $\beta_C = 0$ solution will be discussed further below). If the linear response $(\delta\zeta_{RF})$ is assumed to be a valid approximation for small amplitude RF signals, then it can be determined in terms of the correlation functions

$$\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle \quad \text{and} \quad \left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle \tag{1}$$

as

$$\delta\zeta_{RF} = \sum_{i=1}^{2} A_i \sin(\omega_{rf,i}\tau - \Psi_{rf,i}) = \sum_{i=1}^{2}\left(\frac{a_{rf,i}}{2}\right)\frac{\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle}{\sqrt{\omega_{rf,i}^2 + \left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle^2}}\sin(\omega_{rf,i}\tau - \Psi_{rf,i}), \tag{2}$$

where the RF phase shifts are given by

$$\Psi_{rf,i} = \sin^{-1}\left(\omega_{rf,i} \Big/ \sqrt{\omega_{rf,i}^2 + \left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle^2}\right). \tag{3}$$

It can be seen that as long as the $\langle \cos\cos \rangle$ correlation function is non-zero, the RF phase shift Ψ_{rf} will approach zero as the signal frequency $\omega_{rf,i}$ goes to zero. A useful corollary to this behavior is that as the $\langle \cos\cos \rangle$ correlation function becomes small and of the same order as $\omega_{rf,i}$, the RF phase shift will depend sensitively on β_L or ϕ_{DC} for fixed $\omega_{rf,i}$, or conversely it will be a steep function of $\omega_{rf,i}$ when β_L and ϕ_{DC} are fixed. It will be shown below that the correlation function $\langle \cos\cos \rangle$ becomes small when either β_L or ϕ_{DC} is small.

3. Derivation of the Hamiltonians for Mirror Symmetric Row Arrays

The row arrays that we shall treat here can be considered as a subset of the completely random SQIF array row described by Oppenlander et al.[1] in that they are random but have mirror symmetry about a central junction. The Hamiltonians and equations of motion for mirror symmetric row arrays are described by $N_{cell}+1$ junction phases, along with N_{cell} constraint equations, and N_{cell} circulating currents (or their equivalent fluxes), for a total of $N_{cell}+1$ independent coordinates. The N_{cell} constraints may be included in the Lagrangian along with N_{cell} Lagrange multipliers before forming the Hamiltonian. However, a simpler method is to transform to a set of $N_{cell}+1$ coordinates that include the constraints, which then yield one independent junction phase and N_{cell} independent circulating currents. We illustrate this with an $N_{cell}=4$ array shown below, where the middle junction is allowed to have a critical current that differs from the remaining junctions by a factor of ρ.

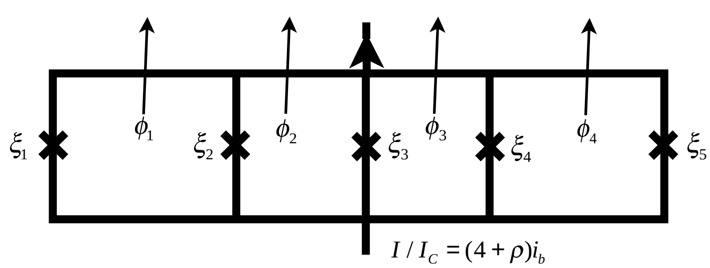


Figure 1. Four-Cell Array

We first transform the external fluxes $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ to the set $\{\phi_a, \phi_b, \phi_c, \phi_s\}$, where $\phi_s = (\phi_1 + \phi_2 + \phi_3 + \phi_4)/4$ would be symmetric in all four fluxes (if the fluxes were equal), while the remaining $N_{cell} - 1$ coordinates are the partly antisymmetric combinations

$$\begin{aligned}
\phi_a &= (\phi_1 + \phi_2 - \phi_3 - \phi_4)/4 \\
\phi_b &= (\phi_1 - \phi_2 - \phi_3 + \phi_4)/4 \\
\phi_c &= (\phi_1 - \phi_2 + \phi_3 - \phi_4)/4.
\end{aligned} \tag{4}$$

The N_{cell} induced fluxes $\eta_k = \beta_{L,kk} i_{circ,k}$ are transformed in an identical way. By writing $\xi_j = \zeta + a_j(\phi_a + \eta_a) + b_j(\phi_b + \eta_b) + c_j(\phi_b + \eta_c) + s_j(\phi_s + \eta_s)$, the constraint equations for the four cells, along with the demand that the coefficients of the four cross terms $\dot{\zeta}\dot{\phi}_i$ in the kinetic energy be zero, can be solved to yield a solution for the coefficients $\{a_i, b_i, c_i, s_i\}$. The transformation of the original junction phases are then given by

$$\begin{aligned}
\xi_1 &= \zeta + 2\frac{(\rho+1)}{(\rho+4)}(\phi_a + \eta_a) + 2(\phi_s + \eta_s) + \frac{2}{(\rho+4)}(\phi_c + \eta_c) \\
\xi_2 &= \zeta + \frac{(\rho-2)}{(\rho+4)}(\phi_a + \eta_a) + (\phi_s + \eta_s) - (\phi_b + \eta_b) - \frac{(\rho+2)}{(\rho+4)}(\phi_c + \eta_c) \\
\xi_3 &= \zeta - \frac{6}{(\rho+4)}(\phi_a + \eta_a) + \frac{2}{(\rho+4)}(\phi_c + \eta_c) \\
\xi_4 &= \zeta + \frac{(\rho-2)}{(\rho+4)}(\phi_a + \eta_a) - (\phi_s + \eta_s) + (\phi_b + \eta_b) - \frac{(\rho+2)}{(\rho+4)}(\phi_c + \eta_c) \\
\xi_5 &= \zeta + 2\frac{(\rho+1)}{(\rho+4)}(\phi_a + \eta_a) - 2(\phi_s + \eta_s) + \frac{2}{(\rho+4)}(\phi_c + \eta_c).
\end{aligned} \tag{5}$$

It can be seen that the phase of the middle junction ξ_3 will have contributions from the fluxes that are antisymmetric (if all cell fluxes are equal) about the center conductor in addition to that from the effective junction phase ζ.

When $\rho = 1$ the effective phase ζ may be obtained by summing all of the junction phases and dividing by five. When $\rho \neq 1$, the expression for ζ will also include flux terms multiplied by a factor proportional to $(\rho - 1)$. The expression for the junction potential energy may be written as

$$
\begin{aligned}
V_{JJ} / E_J &= -\sum_{i=1}^{i=5} \cos(\xi_i) - (\rho - 1)\cos(\xi_3) \\
&= -2\cos[\zeta + 2\tfrac{\rho+1}{\rho+4}(\phi_a + \eta_a) + \tfrac{2}{\rho+4}(\phi_c + \eta_c)]\cos[2(\phi_s + \eta_s)] \\
&\quad - 2\cos[\zeta + \tfrac{\rho-2}{\rho+4}(\phi_a + \eta_a) - \tfrac{\rho+2}{\rho+4}(\phi_c + \eta_c)]\cos[\phi_s + \eta_s) - (\phi_b + \eta_b)] \\
&\quad - \rho\cos[\zeta - \tfrac{6}{\rho+4}(\phi_a + \eta_a) + \tfrac{2}{\rho+4}(\phi_c + \eta_c)].
\end{aligned} \tag{6}
$$

For the case of a two-cell SQUID, the transformed coordinates and junction potential function are given by

$$
\begin{aligned}
\xi_1 &= \zeta + (\phi_s + \eta_s) + \frac{\rho}{(\rho+2)}(\phi_a + \eta_a) \\
\xi_2 &= \zeta - \frac{2}{(\rho+2)}(\phi_a + \eta_a) \\
\xi_3 &= \zeta - (\phi_s + \eta_s) + \frac{\rho}{(\rho+2)}(\phi_a + \eta_a),
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
V_{JJ} / E_J &= -\sum_{i=1}^{i=3} \cos(\xi_i) - (\rho - 1)\cos(\xi_2) \\
&= -2\cos[\zeta + \tfrac{\rho}{\rho+2}(\phi_a + \eta_a)]\cos[(\phi_s + \eta_s)] \\
&\quad - \rho\cos[\zeta - \tfrac{2}{\rho+2}(\phi_a + \eta_a)].
\end{aligned} \tag{8}
$$

For the case of a one-cell SQUID, the transformed coordinates and junction potential energy are given by

$$
\begin{aligned}
\xi_1 &= \zeta + (\phi + \eta)/2 \\
\xi_2 &= \zeta - (\phi + \eta)/2,
\end{aligned} \tag{9}
$$

and

$$
V_{JJ} / E_J = -\sum_{i=1}^{i=2} \cos(\xi_i) = -2\cos(\zeta)\cos[(\phi + \eta)/2]. \tag{10}
$$

4. $\beta_c = 0$ Solution for a Single SQUID

Using the coordinates given above in Eq. 9, the kinetic energy term in the Hamiltonian may be written as

$$T = \frac{C}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 \left\{2\dot{\zeta}^2 + \dot{\eta}_s^2/2\right\}, \tag{11}$$

and in dimensionless units as

$$\frac{T}{E_J} = \frac{M_\zeta}{2}\zeta_{,\tau}^2 + \frac{M_\eta}{2}\eta_{,\tau}^2, \tag{12}$$

with

$$\begin{aligned} &M_\zeta = 2\beta_C, \quad M_\eta = \beta_C/2 \\ &\eta = \beta_L i_{circ}, \\ &E_J = I_C\Phi_0/2\pi, \end{aligned} \tag{13}$$

where

$$\beta_C = C\left(\frac{2\pi}{\Phi_0}\right)I_C R_N^2, \quad \beta_L = L(2\pi I_C/\Phi_0), \quad \tau = \omega_c t, \quad \omega_c = \frac{2\pi}{\Phi_0}I_C R_N. \tag{14}$$

The inductive potential energy may be written as

$$V_\eta = LI_{circ}^2/2 = E_J\beta_L i_{circ}^2/2 = E_J \tfrac{1}{2} M_s\omega_s^2\eta^2 \tag{15}$$

with

$$\omega_s^2 = \frac{2}{\beta_C\beta_L}, \quad i_{circ} = I_{circ}/I_c. \tag{16}$$

The total Hamiltonian, including the forcing current potential, is then given by

$$H = T + V_{JJ} + V_\eta - E_J i_b\zeta, \tag{17}$$

which yields the equations of motion in terms of the transformed coordinates $\{\zeta, \eta\}$ as

$$\begin{aligned} &\beta_c\,\zeta_{,\tau\tau} + \zeta_{,\tau} + \sin(\zeta)\cos[(\phi+\eta)/2] = i_b \\ &\beta_c\,\eta_{,\tau\tau} + \eta_{,\tau} + 2\cos(\zeta)\sin[(\phi+\eta)/2] + \beta_c\omega_\eta^2\eta = 0\ . \end{aligned} \tag{18}$$

For over-damped junctions we set $\beta_C = 0$ and rewrite ζ as $\zeta = \theta + \pi/2$. The equations of motion for θ can then be written in the form used by Swift et al.[1] as

$$\begin{aligned} &\theta,_{\bar{\tau}} = 1 - \cos(\theta)\cos[(\phi+\eta)/2]/i_b = 1 + a\cos(\theta) \\ &\eta,_\tau + (2/\beta_L)\eta = 2\sin(\theta)\sin[(\phi+\eta)/2], \end{aligned} \tag{19}$$

where

$$a = -\cos[(\phi+\eta)/2]/i_b \quad \text{and} \quad \bar{\tau} = \tau i_b. \tag{20}$$

Swift, Strogatz, and Wiesenfeld[2] have shown that for a system of single junctions connected in series, a perturbation theory, based on the exact solution for a single junction, can be developed to determine the DC voltage for the coupled junction system. The set of relative phases for the junctions is not determined by the theory, but there are optimal sets where the stability of the array is maximized. For the SQUID problem, we have extended this theory to determine, in addition to the perturbed frequency $\omega_{DC}[i_b, \phi_{DC}, \beta_L]$, the time dependence of the effective junction phase $\zeta_{DC} = \zeta_0(\omega_{DC} t)$, and the induced flux $\eta_{DC} = \beta_L i_{circ,DC}$ (or equivalently a description in terms of the two junction phases ξ_1 and ξ_2) as functions of i_b, ϕ and β_L.

To develop a perturbation expansion for θ and η, we rewrite the θ equation of motion as

$$\theta,_{\bar{\tau}} = 1 + a_0\cos(\theta) + (a - a_0)\cos(\theta), \quad \text{with} \quad a_0 = -\cos[(\phi + \langle\eta\rangle)/2]/i_b, \tag{21}$$

where

$$\langle\eta\rangle = \frac{1}{T}\int_0^T \eta(\omega\tau)\,d\tau, \quad T = 2\pi/\omega, \quad \eta = \langle\eta\rangle + \delta\eta. \tag{22}$$

Treating the last term as a perturbation, the unperturbed equation of motion for θ may be solved exactly yielding

$$\theta_0(\tau) = 2\tan^{-1}(b_0^{-1}\tan(\varphi_0/2)), \tag{23}$$

where

$$\varphi_0 = \sqrt{1-a_0^2}\,\bar{\tau} = \sqrt{1-a_0^2}\,i_b\tau = \omega_0\tau \quad \text{and} \quad b_0 = \sqrt{\frac{1-a_0}{1+a_0}}. \tag{24}$$

To obtain the first-order correction to the frequency, we use the θ equation of motion to write

$$\frac{\omega}{i_b} = \frac{d\varphi}{d\bar{\tau}} = \frac{d\varphi}{d\theta}\frac{d\theta}{d\bar{\tau}} \approx \frac{\sqrt{1-a_0^2}}{1+a_0\cos(\theta)}\left[1+a_0\cos(\theta)+(a-a_0)\cos(\theta)\right]$$
$$= \sqrt{1-a_0^2} + \frac{\sqrt{1-a_0^2}}{1+a_0\cos(\theta)}(a-a_0)\cos(\theta) = (\omega_0+\delta\omega)/i_b. \tag{25}$$

Using the trigonometric identity[1]

$$\frac{1}{1+a_0\cos(\theta_0)} = \frac{1-a_0\cos(\varphi_0)}{1-a_0^2} \quad \text{and} \quad \cos(\theta_0) = \frac{\cos(\varphi_0)-a_0}{1-a_0\cos(\varphi_0)} \tag{26}$$

the expression for the first-order frequency correction may then be written as

$$\delta\omega = i_b \frac{1}{\sqrt{1-a_0^2}}\frac{1}{2\pi}\int_0^{2\pi}(\cos(\varphi)-a_0)(a(\varphi)-a_0)\,d\varphi. \tag{27}$$

To proceed further it is necessary to solve the equation of motion for η to lowest order, which is obtained by expanding the $\sin[(\phi+\eta)/2]$ term in the equation of motion to first order in η, with solutions given the label η_0

$$\eta,_\tau + (2/\beta_L)\eta = 2\sin(\theta_0)\sin[\tfrac{1}{2}(\phi+\eta)] = 2\sin(\theta_0)\{\sin(\tfrac{1}{2}\phi)(1-\tfrac{1}{8}\eta^2+...)+\cos(\tfrac{1}{2}\phi)(\tfrac{1}{2}\eta-...\}$$
$$\eta_0,_\tau + (2/\beta_L)\eta_0 \approx 2\sin(\theta_0)\{\sin(\tfrac{1}{2}\phi)+\cos(\tfrac{1}{2}\phi)\tfrac{1}{2}\eta_0\}. \tag{28}$$

Using the exact zero-order solution for θ_0, $\sin(\theta_0)$ and $\cos(\theta_0)$ may be evaluated as

$$\sin(\theta_0) = \frac{\sqrt{1-a_0^2}\sin(\omega_0\tau)}{1-a_0\cos(\omega_0\tau)}, \quad \cos(\theta_0) = \frac{\cos(\omega_0\tau)-a_0}{1-a_0\cos(\omega_0\tau)}. \tag{29}$$

This allows the re-expression of the lowest order η_0 equation of motion as

$$\eta_{0,\tau} + f(\tau)[\sin(\phi/2)+\eta_0\cos(\phi/2)] + 2\eta_0/\beta_L = 0, \tag{30}$$

where

$$f(\tau) = \frac{\sin(\omega_0\tau)f_{\sin}}{1-a_0\cos(\omega_0\tau)}, \quad f_{\sin} = -2\sqrt{1-a_0^2}. \tag{31}$$

As a first-order ordinary differential equation, the η_0 equation of motion may be solved exactly as

$$\eta_s^0(\tau) = -e^{-\int_0^{\tau}(2/\beta_L + \cos(\phi/2) f(\tau'))d\tau'} \int_0^{\tau} \sin(\phi/2) f(\tau') e^{\int_0^{\tau'}(2/\beta_L + \cos(\phi/2) f(\tau''))d\tau''} d\tau'. \tag{32}$$

The integral in the two exponential factors may be carried out to yield

$$\int_0^{\tau} f(\tau')d\tau' = \frac{-\sqrt{1-\widehat{a}_0^2}\{\ln[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau)] - i\pi/2\}}{\widehat{a}_0\widehat{\omega}_0}, \tag{33}$$

allowing the solution to be written as

$$\eta_0 = -e^{-2\tau/\beta_L} e^{(\sqrt{1-a_0^2}\cos(\phi/2)/a_0\omega_0)\ln[1-a_0\cos(\omega_0\tau)]} \times$$
$$\int_{-\infty}^{\tau} \frac{e^{2\tau'/\beta_L} e^{(-\sqrt{1-a_0^2}\cos(\phi/2)/a_0\omega_0)\ln[1-a_0\cos(\omega_0\tau')]} f_{\sin}\sin(\phi/2)\sin(\omega_0\tau')}{(1-a_0\cos(\omega_0\tau'))} d\tau'. \tag{34}$$

Due to the fact that

$$\sqrt{1-a_0^2}\cos(\phi/2)/a_0\omega_0) = -1, \tag{35}$$

the denominator in the integrand is canceled out and the integration can be carried out exactly, yielding

$$\eta_0(\tau) = -\left[\frac{2}{i_b}\sin(\phi/2)\right]\left[\frac{\cos(\omega_0\tau) - c\sin(\omega_0\tau)}{(1+c^2)(1-a_0\cos(\omega_0\tau))}\right] = \kappa\frac{\cos(\omega_0\tau) - c\sin(\omega_0\tau)}{1-a_0\cos(\omega_0\tau)}, \tag{36}$$

where

$$c = 2/(\omega_0\beta_L), \quad \kappa = -\left[\frac{2}{i_b}\frac{\sin(\phi/2)}{1+c^2}\right]. \tag{37}$$

The first-order correction to the rotation frequency

$$\delta\omega = i_b\frac{1}{\sqrt{1-a_0^2}}\frac{\omega_0}{2\pi}\int_0^{2\pi}(\cos(\omega_0\tau) - a_0)[a(\varphi_0) - a_0(\varphi_0)]d\tau, \tag{38}$$

may now be carried out by expanding $a(\varphi_0)$ in terms of powers of η_0. The integration required after expanding

$$a(\varphi_0) = -\cos[\tfrac{1}{2}(\phi + \eta_0(\tau))]/i_b \tag{39}$$

to any desired order in η_0, may be carried out in closed form. To second order in η_0 this yields

$$\delta\omega = -\sin(\phi/2)\langle\eta\rangle(a_0 - \frac{1}{a_0})(\frac{1}{2i_b})$$
$$- \cos(\phi/2)(\frac{1}{8i_b})\left\{\langle\eta^2\rangle(a_0 - \frac{1}{a_0}) + \frac{\kappa^2}{a_0}\left[\frac{1}{\sqrt{1-a_0^2}} - (1+c^2)\frac{1+(a_0-1)/b}{a_0^2}\right]\right\}, \quad (40)$$

where

$$\langle\eta\rangle = -\kappa\frac{1+a_0-b_0^{-1}}{a_0(1+a_0)} \quad (41)$$

and

$$\langle\eta^2\rangle = \kappa^2\frac{1}{(1+a_0)}\left\{\frac{b_0^{-1}}{1-a_0^2} + \frac{(1+c^2)(1+a_0-b_0^{-1})}{a_0^2}\right\}. \quad (42)$$

Making the replacement

$$\omega_0 \to \omega_{DC} = \omega_0 + \delta\omega \quad (43)$$

in the expressions for η_0 and θ_0 then yields the first order unperturbed solutions η_{DC} and θ_{DC}.

The unperturbed flux η_{DC} can also be expressed in terms of ζ_0 as

$$\eta_{DC} = \frac{-2\sin(\phi_{DC}/2)}{i_b(1+c^2)(1-a_0^2)}\left[a_0 - \sqrt{1+c^2(1-a_0^2)}\sin(\zeta_0 - (\pi/2+\psi))\right]$$
$$= p + q\sin(\zeta_0 - (\pi/2+\psi)), \quad (44)$$

where the phase shift ψ is given by

$$\psi = \sin^{-1}\left[(i_b\beta_L/2)/\sqrt{1+(i_b\beta_L/2)^2}\right]. \quad (45)$$

In this form, η_{DC} has both a constant and an oscillating component, and the result can be seen to make sense physically by considering the supercurrent $I_C\sin(\zeta)$ going through the effective ζ junction. This current amplitude will be a maximum when $\zeta = \pm\pi/2$, at which point η_{DC} will be at a minimum (for $\psi = 0$) and vice versa. As the SQUID inductance increases from zero, the oscillating part of η_{DC} will be shifted in phase by an amount that is a function of the flux $i_b\beta_L$.

Below is a plot showing η_{DC} (black) and η_{DC} obtained by a numerical solution (green) of the coupled ODEs for $i_b = 1.05$, $\phi_{DC} = 0.2\pi$ and $\beta_L = 1.0$, where the motion is markedly non-sinusoidal. It can be seen that the value obtained for the frequency is fairly accurate, while the analytic expression for the amplitude of η_{DC} differs from the amplitude of the numerical solution by about one to two percent.

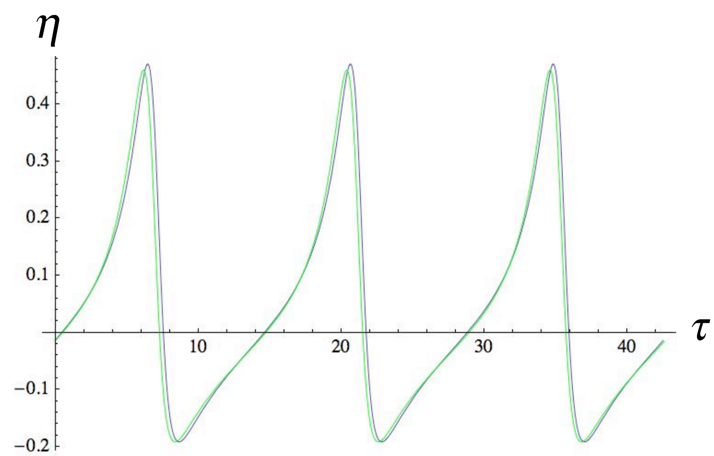


Figure 2. SQUID-Induced Flux $\eta(\tau)$ vs τ

5. *Linear Response of a SQUID to an RF Signal for the $\beta_c = 0$ Case*

To determine the linear response of a SQUID to an rf signal, we write the equations of motion for the effective phase ζ

$$\beta_c \zeta,_{\tau\tau} + \zeta,_{\tau} + \sin[\zeta]\cos[(\phi+\eta)/2] = i_b \tag{46}$$

in terms of the unperturbed solution $\{\zeta_{DC}, \eta_{DC}\}$ and denote the linear response to the RF signal as $\delta\zeta$, where the RF signal is given by $\delta\phi = A\sin(\omega_{rf}\tau)$. Rewriting the equations of motion with

$$\zeta = \zeta_{DC} + \delta\zeta_{rf} \quad \text{and} \quad \phi = \phi_{\mathrm{DC}} + A\sin(\omega_{rf}\tau) \tag{47}$$

then yields

$$\delta\zeta_{rf},_{\tau} + \cos[\zeta_{DC}]\cos[(\phi_{DC}+\eta_{DC})/2]\delta\zeta_{rf} - \sin[\zeta_{DC}]\sin[(\phi_{DC}+\eta_{DC})/2]\tfrac{A}{2}\sin(\omega_{rf}\tau) = 0, \tag{48}$$

where we have subtracted out the DC solution and kept only linear terms in $\delta\zeta$ and A. Assuming that the frequency of the RF signal is much less than ω_{DC}, it is reasonable to average over the rapid oscillations of ζ_{DC} and η_{DC} which yields

$$\delta\zeta_{rf},_{\tau} + \langle\cos[\zeta_{DC}]\cos[(\phi_{DC}+\eta_{DC})/2]\rangle\delta\zeta_{rf} - \langle\sin[\zeta_{DC}]\sin[(\phi_{DC}+\eta_{DC})/2]\rangle\tfrac{A}{2}\sin(\omega_{rf}\tau) = 0. \tag{49}$$

The solution for the linear response $\delta\zeta_{RF}$ can be written in terms of the two correlation functions as

$$\delta\zeta_{rf} = \frac{A}{2}\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle \frac{\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle \sin(\omega_{rf}\tau)-\omega_{rf}\cos(\omega_{rf}\tau)}{\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle^2+\omega_{rf}^2}. \tag{50}$$

This may be re-expressed in the simpler form of a single sinusoid as

$$\delta\zeta_{rf} = \frac{A}{2}\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle \frac{\sin(\omega_{rf}\tau-\Psi_{rfi})}{\sqrt{\omega_{rf}^2+\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle^2}}, \tag{51}$$

where the RF phase shifts is given by

$$\Psi_{rf} = \sin^{-1}\left(\omega_{rf} \Big/ \sqrt{\omega_{rf}^2+\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle^2}\right). \tag{52}$$

It can be easily seen that Ψ_{rf} will approach zero as the signal frequency ω_{rf} goes to zero. A useful corollary to this behavior is that as the $\langle\cos\cos\rangle$ correlation function becomes small and of the same order as ω_{rf}, the RF phase shift will depend sensitively on β_L or ϕ_{DC} for fixed ω_{rf}, or conversely it will be a steep function of ω_{rf} when β_L and ϕ_{DC} are fixed.

By using the expressions

$$\cos(\zeta_{DC}) = -\frac{\sqrt{1-a_{DC}^2}\,\sin(\omega_{DC}\,\tau)}{1-a_{DC}\cos(\omega_{DC}\,\tau)},\quad \sin(\zeta_{DC}) = \frac{\cos(\omega_{DC}\,\tau)-a_{DC}}{1-a_{DC}\cos(\omega_{DC}\,\tau)}, \tag{53}$$

the correlation functions can expanded to any order in $(\eta_{DC})^n$ and integrated term by term. Thus to second order in η_{DC} we have

$$\begin{aligned}\left\langle \cos[\zeta_{DC}]\cos[(\phi_{DC}+\eta_{DC})/2]\right\rangle &= \frac{\omega_{DC}}{2\pi}\int_0^{2\pi/\omega_{DC}} \cos[\zeta_{DC}(\tau)]\cos[(\phi_{DC}+\eta_{DC}(\tau))/2]\,d\tau \\ &= -\frac{\omega_{DC}}{i_b}\left\{-\frac{\sin(\phi_s/2)}{2}\kappa c\frac{(1+a_{DC}-b_{DC}^{-1})}{a_{DC}^2(1-a_{DC})}+\frac{\cos(\phi_s/2)}{4}\kappa^2 c\frac{\{1-b_{DC}^{-1}+a_{DC}[1-a_{DC}(1+a_{DC}-\frac{3}{2}b_{DC}^{-1})]\}}{(1-a_{DC})a_{DC}(1+a_{DC})^2}\right\}\end{aligned} \tag{54}$$

and

(55)

$$\left\langle \sin[\zeta_{DC}]\sin[(\phi_{DC}+\eta_{DC})/2]\right\rangle = \frac{\omega_{DC}}{2\pi}\int_0^{2\pi/\omega_{DC}} \sin[\zeta_{DC}(\tau)]\sin[(\phi_{DC}+\eta_{DC}(\tau))/2]d\tau$$

$$= \cos(\phi_s/2)\sin(\phi_s/2)I_1 + (\cos(\phi_s/2)^2 + \cos(\phi_s/2)-1)\frac{\kappa}{2}I_2 - a_{DC}[\cos(\phi_s/2) - \tfrac{1}{2}\sin(\phi_s/2)^2]\frac{\kappa}{2}I_3$$

$$-\sin(\phi_s/2)\cos(\phi_s/2)\frac{\kappa^2}{8}(1-c^2)I_4 + a_{DC}\sin(\phi_s/2)\cos(\phi_s/2)\frac{\kappa^2}{8}I_5 - \sin(\phi_s/2)\cos(\phi_s/2)\frac{\kappa^2}{8}c^2 I_6.$$

The expressions for the integrals I_1 through I_6 are given in appendix A.

Plots of the two correlation functions $\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$ and $\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$, expanded to second order in η_{DC}, are plotted below for $i_b = 1.05$ and $i_b = 4$ (n.b. vertical scales differ).

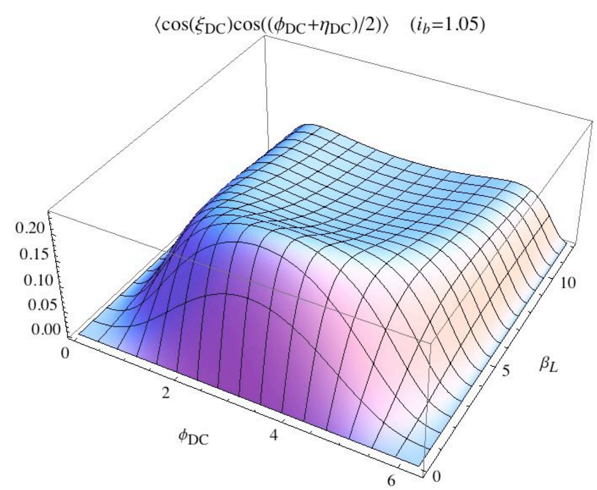


Figure 3.

$\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$
vs. ϕ_{DC} and β_L with $i_b = 1.05$

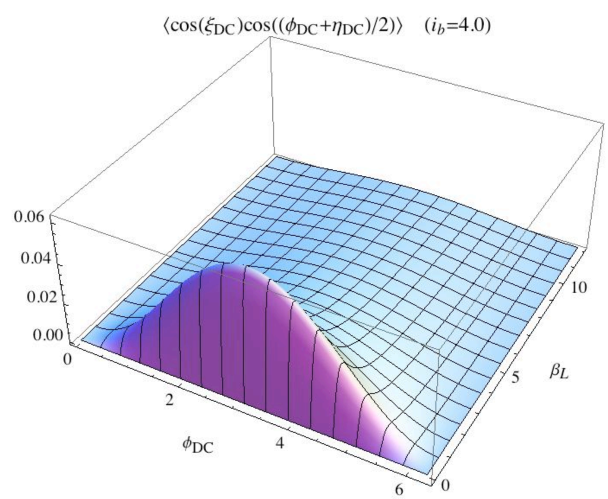
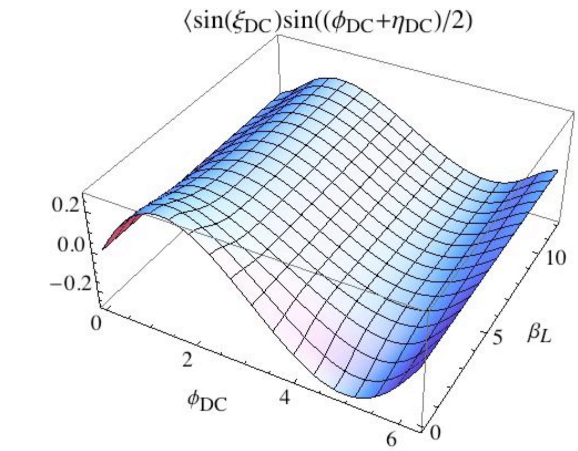


Figure 4.

$\left\langle \cos(\zeta_{DC})\cos(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$
vs. ϕ_{DC} and β_L with $i_b = 4.0$



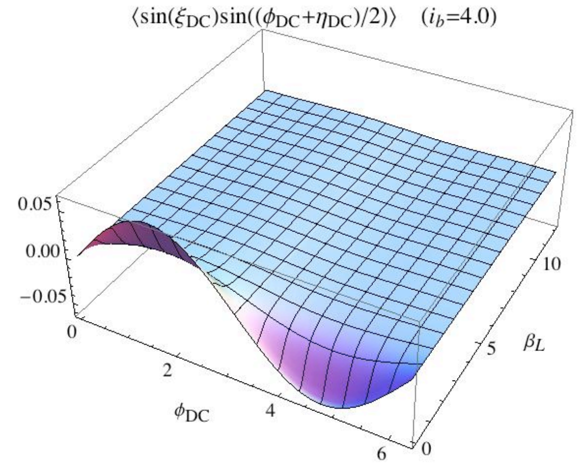
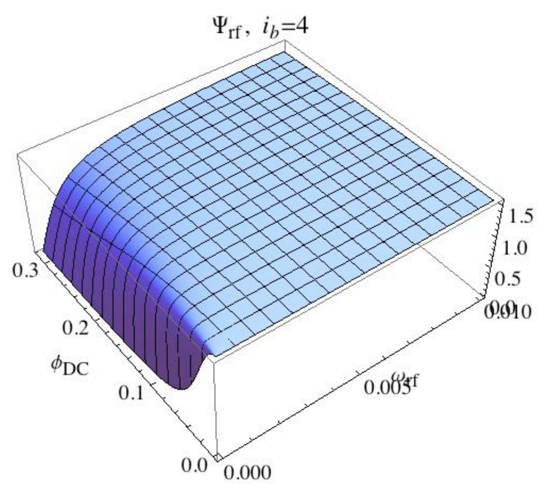


Figure 5.
$\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$ vs. ϕ_{DC} and β_L , $i_b = 1.05$

Figure 6.
$\left\langle \sin(\zeta_{DC})\sin(\frac{\phi_{DC}+\eta_{DC}}{2})\right\rangle$ vs. ϕ_{DC} and β_L , $i_b = 4.0$

It can be seen that for small values of ϕ_{DC} or β_L for any value of i_b, or for large β_L and large i_b, that $\langle \cos\cos \rangle$ will be small, allowing the RF phase shift to be sensitive to small changes in frequency. We have plotted the RF phase shift below for $\beta_L = 5.0$, for signal frequencies from $\omega_{rf} = 0$ to $\omega_{rf} = 0.01\omega_c$. It can be seen that Ψ_{rf} is a very steep function of ω_{rf} for small values of ϕ_{DC} for the chosen SQUID parameters ($i_b = 4$ for both plots).



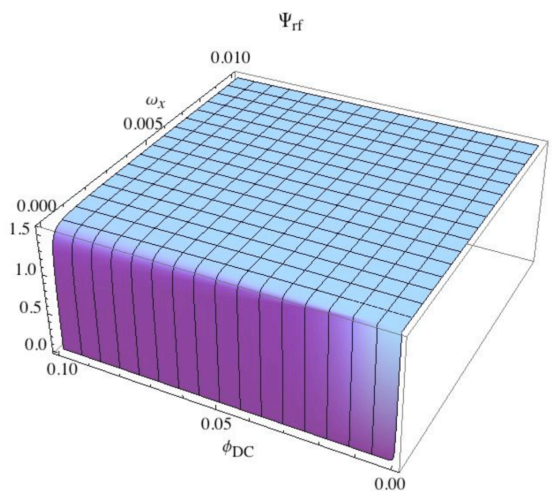


Figure 7.
Two Cell SQUID Phase Ψ_{rf} vs ϕ_{DC} and ω_{rf}

Figure 8.
Two Cell SQUID Phase Ψ_{rf} vs ϕ_{DC} and ω_{rf}

6. $\beta_c = 0$ Solution for a Two-Cell SQUID

For external magnetic fluxes near $\phi_a \sim 0, \phi_s \sim 0$, the equations of motion for the two-cell SQUID can be solved using the perturbation method developed above for the single cell SQUID. From the numerical solution for the two-cell case, it can be seen, however, that using the approximate solution of Eq. 23 for the effective ζ junction (which consists of a string of single pulses), will not be valid when the η_a and η_s amplitudes become large enough to markedly affect the ζ junction, as shown in the example below.

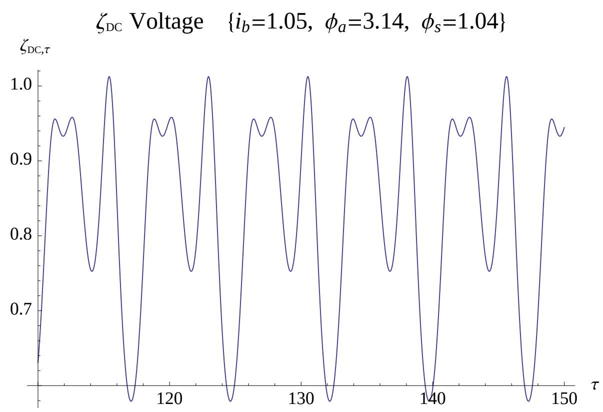


Figure 9.
Two-Cell SQUID Voltage $d\zeta_{DC}(\tau)/d\tau$ vs. τ

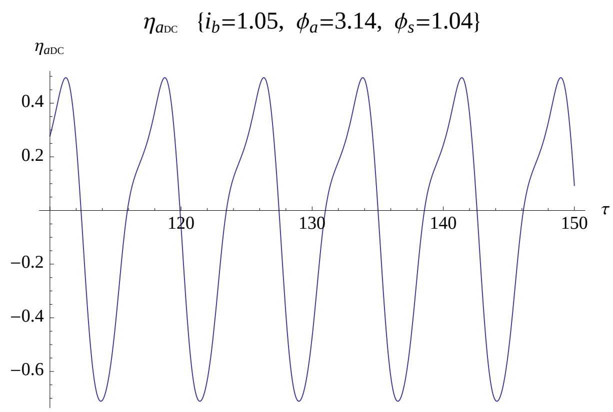


Figure 10.
Two-Cell SQUID Induced Flux $\eta_{a_{DC}}(\tau)$ vs. τ

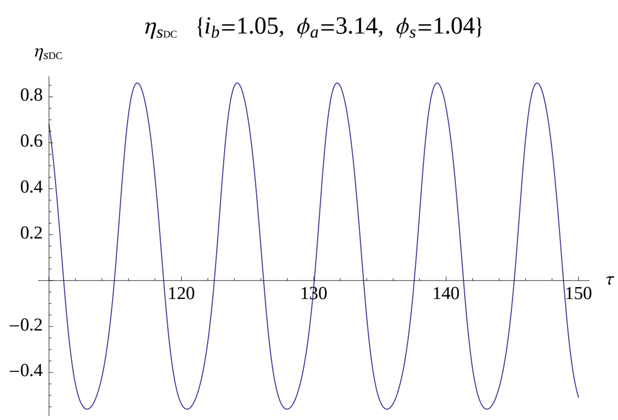


Figure 11.
Two-Cell SQUID Induced Flux $\eta_{s_{DC}}(\tau)$ vs. τ

To cover the entire range for ϕ_a and ϕ_s it will be necessary to include the time-dependent η_a and η_s solutions in determining $\zeta(\tau)$ and perhaps an iteration of the coupled solutions. In what follows, we develop the solutions for the case where ϕ_a and ϕ_s are near zero.

Using the coordinates given above in Eq. **7**, the kinetic energy term in the Hamiltonian may be written as

$$T = \frac{C}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 \left\{(2+\rho)\dot{\zeta}^2 + 2\dot{\eta}_s^2 + \frac{2\rho}{2+\rho}\dot{\eta}_a^2\right\}, \tag{56}$$

and in dimensionless units as

$$\frac{T}{E_J} = \frac{M_\zeta}{2}\zeta_{,\tau}^2 + \frac{M_s}{2}\eta_{s,\tau}^2 + \frac{M_a}{2}\eta_{a,\tau}^2, \tag{57}$$

with

$$\begin{aligned} &M_\zeta = (2+\rho)\beta_C, \quad M_s = 2\beta_C, \quad M_a = \frac{2\rho}{2+\rho}\beta_C \\ &\eta_s = \beta_{Ls} i_s, \quad \eta_a = \beta_{La} i_a, \\ &i_s = (i_1 + i_2)/2, \quad i_a = (i_1 - i_2)/2, \\ &\beta_{Ls} = \beta_L + \beta_M, \quad \beta_{La} = \beta_L - \beta_M. \end{aligned} \tag{58}$$

The inductive potential energy may be written as

$$V_\eta = \sum_{k,l=1}^{2} L_{k.l} I_k I_l = E_J \sum_{k,l=1}^{2} (\beta_L)_{k,l} i_k i_l = E_J\left[\tfrac{1}{2} M_s \omega_s^2 \eta_s^2 + \tfrac{1}{2} M_a \omega_a^2 \eta_a^2\right] \tag{59}$$

with

$$\omega_s^2 = \frac{1}{\beta_C \beta_{Ls}}, \quad \omega_a^2 = \frac{(2+\rho)}{\rho}\frac{1}{\beta_C \beta_{La}}. \tag{60}$$

The total Hamiltonian, including the forcing current potential, is given by

$$H = T + V_{JJ} + V_\eta - E_J (2+\rho) i_b \zeta, \tag{61}$$

which yields (using V_{JJ} given in Eq. **10**) the equations of motion

$$\begin{aligned} M_\zeta \zeta_{,\tau\tau} + (2+\rho)\zeta_{,\tau} + 2\sin[\zeta + \tfrac{\rho}{2+\rho}(\phi_a + \eta_a)]\cos(\phi_s + \eta_s) \\ + \rho \sin[\zeta - \tfrac{2}{2+\rho}(\phi_a + \eta_a)] = (2+\rho) i_b, \end{aligned} \tag{62}$$

$$M_s \eta_{s,\tau\tau} + 2\eta_{s,\tau} + 2\cos[\zeta + \tfrac{\rho}{2+\rho}(\phi_a + \eta_a)]\sin(\phi_s + \eta_s) + M_s\omega_s^2 \eta_s = 0, \tag{63}$$

and

$$\begin{aligned} M_a \eta_{a,\tau\tau} &+ \tfrac{2\rho}{2+\rho}\eta_{a,\tau} + \tfrac{2\rho}{2+\rho}\sin[\zeta + \tfrac{\rho}{2+\rho}(\phi_a + \eta_a)]\cos(\phi_s + \eta_s) \\ &- \tfrac{2\rho}{2+\rho}\sin[\zeta - \tfrac{2}{2+\rho}(\phi_a + \eta_a)] + M_a\omega_a^2 \eta_a = 0. \end{aligned} \tag{64}$$

7. Solution for the ζ Equation of Motion for a Two-Cell SQUID with $\beta_C = 0$

The equation of motion for ζ may be solved in essentially the same manner as for the single-cell SQUID case. The two sin terms are first expanded and regrouped to yield

$$M_\zeta \zeta_{,\tau\tau} + (2+\rho)\zeta_{,\tau} + 2[\sin(\zeta)a_\zeta + \cos(\zeta)b_\zeta] = (2+\rho)i_b, \tag{65}$$

with

$$\begin{aligned} a_\zeta &= \cos[\tfrac{\rho}{2+\rho}(\phi_a + \eta_a)]\cos(\phi_s + \eta_s) + \tfrac{\rho}{2}\cos[\tfrac{2}{2+\rho}(\phi_a + \eta_a)] \\ b_\zeta &= \sin[\tfrac{\rho}{2+\rho}(\phi_a + \eta_a)]\cos(\phi_s + \eta_s) - \tfrac{\rho}{2}\sin[\tfrac{2}{2+\rho}(\phi_a + \eta_a)]. \end{aligned} \tag{66}$$

Writing

$$\sin(\zeta)a_\zeta + \cos(\zeta)b_\zeta = [\sin(\zeta)\cos(\gamma) + \cos(\zeta)\sin(\gamma)]\sqrt{a_\zeta^2 + b_\zeta^2} = \sin(\zeta + \gamma_\zeta)c_\zeta, \tag{67}$$

where

$$\gamma_\zeta = \tan^{-1}(b_\zeta / a_\zeta), \quad c_\zeta = \sqrt{a_\zeta^2 + b_\zeta^2}, \tag{68}$$

the lowest order equation of motion for ζ may then be re-expressed with $\gamma = \langle \gamma_\zeta \rangle$ as

$$\beta_C (\zeta + \gamma)_{,\tau\tau} + (\zeta + \gamma)_{,\tau} + (2c_\zeta / (2+\rho))\sin(\zeta + \gamma) = i_b. \tag{69}$$

Changing to a new coordinate

$$\mathcal{Z} = \zeta + \gamma \tag{70}$$

then yields an equation of motion equivalent to that for a single Josephson junction.
Setting

$$\hat{a} = -\frac{2c_\zeta}{i_b(2+\rho)} \tag{71}$$

allows the equations of motion for $\mathcal{Z} = \theta + \pi/2$ for the $\beta_C = 0$ case to be cast in exactly the same form as given above in Eq. 18.

Shown below are surface plots of the critical current $\frac{2}{2+\rho}c_{\zeta}$ vs ϕ_s and ϕ_a for various values of ρ, and vs ρ for the-two cell SQUID

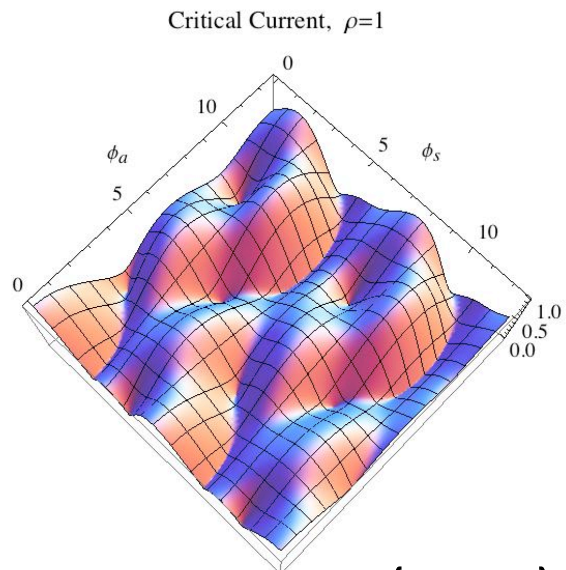


Figure 12. Critical Current vs. $\left\{\phi_a, \phi_{s,}\ \rho = 1\right\}$

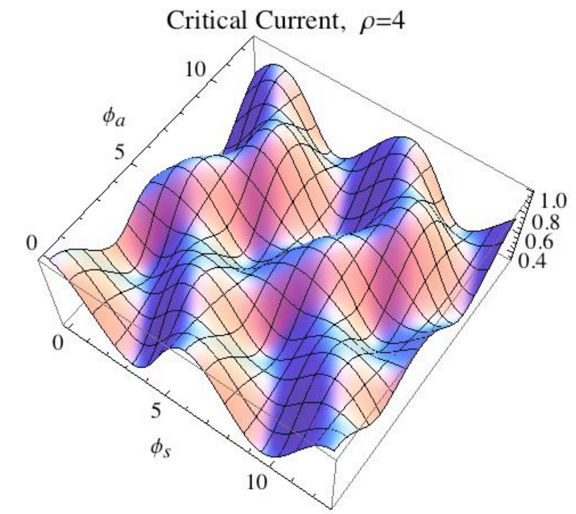


Figure 13. Critical Current vs. $\left\{\phi_a, \phi_{s,}\ \rho = 4\right\}$

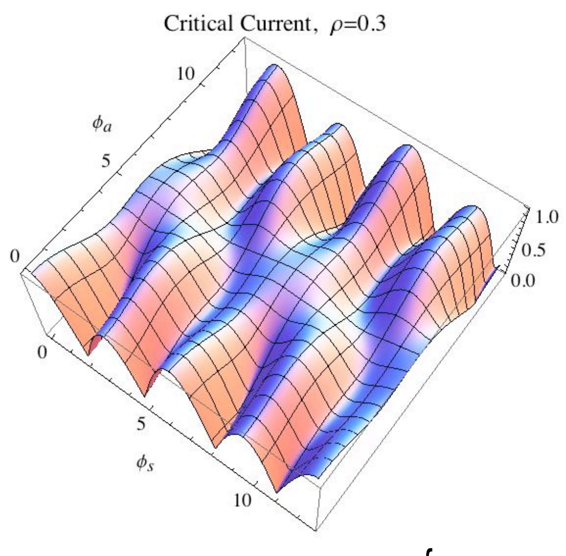


Figure 14. Critical Current vs. $\left\{\phi_a, \phi_{s,}\ \rho = 0.3\right\}$

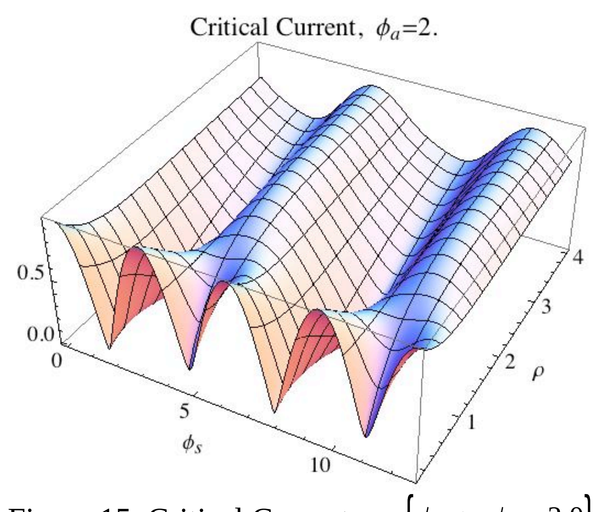


Figure 15. Critical Current vs. $\left\{\phi_{s,}\ \rho,\ \ \phi_a = 2.0\right\}$

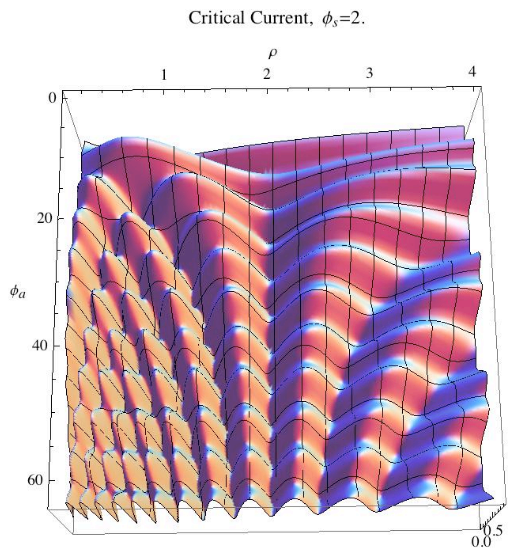


Figure 16. Critical Current vs. $\left\{\phi_a, \rho, \;\phi_{s,} = 2.0\right\}$

8. $\beta_C = 0$ Solution for the η_s Equation of Motion

To derive the η_s equation of motion, we first re-express the $\cos(\zeta_0)$ and $\sin(\zeta_0)$ terms as

$$\begin{aligned}
\cos(\zeta_0) &= \cos(\mathcal{Z}_0 - \gamma) = \cos(\mathcal{Z}_0)\cos(\gamma) + \sin(\mathcal{Z}_0)\sin(\gamma) \\
&= \frac{-\sqrt{1-\hat{a}_0^2}\,\sin(\hat{\omega}_0\,\tau)\cos(\gamma) + (\cos(\hat{\omega}_0\,\tau) - \hat{a}_0)\sin(\gamma)}{1-\hat{a}_0\cos(\hat{\omega}_0\,\tau)} \\
\sin(\zeta_0) &= \sin(\mathcal{Z}_0 - \gamma) = \sin(\mathcal{Z}_0)\cos(\gamma) - \cos(\mathcal{Z}_0)\sin(\gamma) \\
&= \frac{(\cos(\hat{\omega}_0\,\tau) - \hat{a}_0)\cos(\gamma) + \sqrt{1-\hat{a}_0^2}\,\sin(\hat{\omega}_0\,\tau)\sin(\gamma)}{1-\hat{a}_0\cos(\hat{\omega}_0\,\tau)}.
\end{aligned} \tag{72}$$

To obtain the equation of motion to first order in η_s we first approximate $\eta_a \to \langle \eta_a \rangle \approx 0$ to get

$$M_s\,\eta_{s,\tau\tau} + 2\,\eta_{s,\tau} + 2\cos(\zeta + \tfrac{\rho}{2+\rho}\phi_a)\sin(\phi_s + \eta_s) + M_s\omega_s^2\,\eta_s = 0. \tag{73}$$

The $\beta_C = 0$ equation of motion to first order in η_s may then be written using the results of Eq. 53 as

$$\eta^0_{s,\tau} + f(\tau)[\sin(\phi_s) + \eta^0_s\cos(\phi_s)] + \eta^0_s / \beta_{Ls} = 0, \tag{74}$$

where

$$f(\tau) = \frac{\sin(\widehat{w}_0\tau) f_{\sin}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)} + \frac{(\cos(\widehat{w}_0\tau)-\widehat{a}_0) f_{\cos}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)},$$
$$f_{\sin} = -\sqrt{1-\widehat{a}_0^2}\cos(\gamma - k\phi_a), \qquad (75)$$
$$f_{\cos} = \sin(\gamma - k\phi_a),$$

and

$$k = \frac{\rho}{2+\rho}. \qquad (76)$$

The solution η_s^0 of this first-order ODE can be written as

$$\eta_s^0(\tau) = -e^{-\int_0^{\tau}(1/\beta_{Ls}+\cos(\phi_s)f(\tau'))d\tau'}\int_0^{\tau}\sin(\phi_s)f(\tau')e^{\int_0^{\tau'}(1/\beta_{Ls}+\cos(\phi_s)f(\tau''))d\tau''}d\tau'. \qquad (77)$$

The integral in the two exponential factors may be carried out to yield

$$\int_0^{\tau} f(\tau')d\tau' = \frac{-\sin(\gamma-k\phi_a)(\widehat{\omega}_0\tau) - \sqrt{1-\widehat{a}_0^2}\cos(\gamma-k\phi_a)\{\ln[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau)] - i\pi/2\}}{\widehat{a}_0\widehat{\omega}_0}$$
$$+\frac{\sin(\gamma-k\phi_a)(1-\widehat{a}_0^2)2\tan^{-1}[\tan(\widehat{\omega}_0\tau/2)/\widehat{b}]}{\widehat{a}_0\widehat{\omega}_0}. \qquad (78)$$

The $\tan^{-1}$ term above may be re-expressed as $2\tan^{-1}[\tan(\widehat{\omega}_0\tau/2)/\widehat{b}] = \widehat{\theta}_0(\tau)$ and approximated by $\widehat{\theta}_0(\tau) \approx \widehat{\omega}_0\tau$. With this approximation the solution can be written as

$$\eta_s^0(\tau) = e^{-\Omega_s\tau}\frac{-1}{[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau')]^{1-\lambda_s}}\int_0^{\tau}\frac{e^{\Omega_s\tau'}\sin(\phi_s)f(\tau')}{[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau')]^{\lambda_s-1}}d\tau', \qquad (79)$$

where

$$\Omega_s = 1/\beta_{Ls} + \cos(\phi_s)\sin(\gamma-k\phi_a)(\sqrt{1-\widehat{a}_0^2}-1)/\widehat{a}_0, \qquad (80)$$

and

$$\lambda_s = 1 + \frac{\cos(\phi_s)\cos(\gamma-k\phi_a)}{\widehat{a}_0 i_b} = 1 - \frac{(2+\rho)\cos(\phi_s)\cos(\gamma-k\phi_a)}{2\cos(\phi_s)+\rho}. \qquad (81)$$

The final integration required for the η_s^0 solution can be carried out by expanding the denominator in the integrand as a series in $\widehat{a}_0$ and can be written as (82)

$$\eta_s^0(\tau) = \frac{-\sin(\phi_s^0)}{\widehat{\omega}_0\widehat{a}_0(1-\lambda_s)}\left\{f_{\sin}\left[1-\Omega_s\frac{F(\tau,\lambda_s-1,\Omega_s)}{\left[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau)\right]^{1-\lambda_s}}\right] + f_{\cos}\left[\frac{(1-\widehat{a}_0^2)F(\tau,\lambda_s,\Omega_s) - F(\tau,\lambda_s-1,\Omega_s)}{\widehat{a}_0\left[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau)\right]^{1-\lambda_s}}\right]\right\},$$

where the evaluation of

$$F(\tau,\lambda,\Omega)=\int_{-\infty}^{\tau}\frac{e^{\Omega(\tau'-\tau)}d\tau'}{\left[1-\hat{a}_0\cos(\hat{\omega}_0\tau')\right]^{\lambda}}, \tag{83}$$

in terms of hypergeometric functions, is presented in Appendix B.

The integration required for the η_s^0 solution can be carried out in closed form if $\hat{\lambda}_s=0$. This will be true if ϕ_a and ϕ_s are near 0. With this assumption η_s^0 can be written as

$$\eta_s^0(\tau)=e_0+\frac{e_c\cos(\hat{\omega}_0\tau)+e_s\sin(\hat{\omega}_0\tau)}{1-\hat{a}_0\cos(\hat{\omega}_0\tau)}, \tag{84}$$

where

$$\begin{aligned}
e_0 &= \sin(\phi_s)\sin(\gamma-k\phi_a)\hat{a}_0/\hat{\Omega}\\
e_c &= -\sin(\phi_s)[\sqrt{1-\hat{a}_0^2}\cos(\gamma-k\phi_a)\hat{\omega}_0+\sin(\gamma-k\phi_a)\hat{\Omega}]/(\hat{\omega}_0^2+\hat{\Omega}^2)\\
e_s &= -\sin(\phi_s)[\sin(\gamma-k\phi_a)\hat{\omega}_0-\sqrt{1-\hat{a}_0^2}\cos(\gamma-k\phi_a)\hat{\Omega}]/(\hat{\omega}_0^2+\hat{\Omega}^2).
\end{aligned} \tag{85}$$

In the limit of $\gamma\to 0$, this expression for η_s^0 has the same form as that for a single-cell SQUID if the doubled cell area is taken into account, i.e.,

$$\left\{\frac{\eta_0}{2},\frac{\phi_s}{2},\frac{\beta_L}{2}\right\}\to\{\eta_s,\phi_s,\beta_{Ls}\}. \tag{86}$$

9. $\beta_C=0$ Solution for the η_a Equation of Motion

To obtain the equation of motion to first order in η_a we first approximate

$$\cos(\phi_s+\eta_s)\equiv\cos(\phi_s^+)=\cos(\phi_s)\cos(\eta_s)-\sin(\phi_s)\sin(\eta_s)\approx\cos(\phi_s)(1-\left\langle\tfrac{1}{2}\eta_s^2\right\rangle)-\sin(\phi_s)\left\langle\eta_s\right\rangle \tag{87}$$

to get

$$\begin{aligned}
M_a\eta_{a,\tau\tau}+\tfrac{2\rho}{2+\rho}\eta_{a,\tau}+\tfrac{2\rho}{2+\rho}\sin[\zeta+\tfrac{\rho}{2+\rho}(\phi_a+\eta_a)]\cos(\phi_s^+)&\\
-\tfrac{2\rho}{2+\rho}\sin[\zeta-\tfrac{2}{2+\rho}(\phi_a+\eta_a)]+M_a\omega_a^2\eta_a=0.&
\end{aligned} \tag{87}$$

The $\beta_C=0$ equation of motion to first order in η_a may then be written as

$$\eta_{a,\tau}^0+\sin(\zeta_0)\left\{a_x+a_{xx}\eta_a^0\right\}+\cos(\zeta_0)\left\{b_x+b_{xx}\eta_a^0\right\}+\tfrac{2+\rho}{\rho\beta_{La}}\eta_a^0=0 \tag{88}$$

where

$$
\begin{aligned}
a_x &= \cos(k\phi_a)\cos(\phi_s^+) - \cos(K\phi_a) \\
b_x &= \sin(k\phi_a)\cos(\phi_s^+) + \sin(K\phi_a) \\
a_{xx} &= -k\sin(k\phi_a)\cos(\phi_s^+) + K\sin(K\phi_a) \\
b_{xx} &= k\cos(k\phi_a)\cos(\phi_s^+) + K\cos(K\phi_a)
\end{aligned}
\tag{89}
$$

and where we have defined

$$
k = \frac{\rho}{2+\rho}, \quad \text{and} \quad K = \frac{2}{2+\rho}. \tag{90}
$$

With the η_a^0 equation of motion rewritten as

$$
\eta_{a,\tau}^0 + f(\tau)\eta_a^0 + g(\tau) + (\tfrac{1}{k\beta_{La}})\eta_a^0 = 0 \tag{91}
$$

a solution can then be obtained as

$$
\eta_a^0(\tau) = -e^{-\int_0^{\tau}[(2+\rho)/\beta_{La} + f(\tau')]d\tau'} \int_0^{\tau} e^{\int_0^{\tau'}[(2+\rho)/\beta_{La} + f(\tau'')]d\tau''} g(\tau')d\tau', \tag{92}
$$

where the functions $f(\tau)$ and $g(\tau)$ are given by

$$
\begin{aligned}
f(\tau) &= \frac{\sin(\widehat{w}_0\tau) f_{\sin}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)} + \frac{(\cos(\widehat{w}_0\tau) - \widehat{a}_0) f_{\cos}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)} \\
f_{\sin} &= -\sqrt{1-\widehat{a}_0^2}\left(k\cos(\gamma - k\phi_a)\cos(\phi_s^+) + K\cos(\gamma + K\phi_a)\right) \\
f_{\cos} &= k\sin(\gamma - k\phi_a)\cos(\phi_s^+) + K\sin(\gamma + K\phi_a)
\end{aligned}
\tag{93}
$$

and

$$
\begin{aligned}
g(\tau) &= \frac{\sin(\widehat{w}_0\tau) g_{\sin}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)} + \frac{(\cos(\widehat{w}_0\tau) - \widehat{a}_0) g_{\cos}}{1-\widehat{a}_0\cos(\widehat{w}_0\tau)} \\
g_{\sin} &= \sqrt{1-\widehat{a}_0^2}\left(\sin(\gamma - k\phi_a)\cos(\phi_s^+) - \sin(\gamma + K\phi_a)\right) \\
g_{\cos} &= \left(\cos(\gamma - k\phi_a)\cos(\phi_s^+) - \cos(\gamma + K\phi_a)\right)
\end{aligned}
\tag{94}
$$

The integral in the two exponential factors may be carried out as in the η_s^0 case above to yield

$$
\eta_a^0(\tau) = -e^{-\Omega_a\tau}\frac{1}{[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau')]^{1-\lambda_a}}\int_0^{\tau}\frac{e^{\Omega_a\tau'}\ h(\tau')}{[1-\widehat{a}_0\cos(\widehat{\omega}_0\tau')]^{\lambda_a - 1}}d\tau', \tag{95}
$$

where

$$
\Omega_a = (2+\rho)/\beta_{La} + \left\{k\sin(\gamma - k\phi_a)\cos(\phi_s^+) + K\sin(\gamma + K\phi_a)\right\}\frac{\sqrt{1-\widehat{a}_0^2} - 1)}{\widehat{a}_0} \tag{96}
$$

and

$$\lambda_a = 1 - \frac{k\cos(\gamma - k\phi_a)\cos(\phi_s^+) + K\cos(\gamma + K\phi_a)}{\cos(\phi_s^+)}. \tag{97}$$

The final integration required for the η_a^0 solution can be carried out by expanding the denominator in the integrand as a series in $\hat{a}_0$ and can be written as

$$\eta_a^0(\tau) = -\left\{ \frac{g_{\sin}}{\hat{\omega}_0 \hat{a}_0 (1-\lambda_a)} \left[1 - \Omega_a \frac{F(\tau, \lambda_a - 1, \Omega_a)}{\left[1 - \hat{a}_0 \cos(\hat{\omega}_0 \tau)\right]^{1-\lambda_a}} \right] + \frac{g_{\cos}}{\hat{a}_0} \left[\frac{(1-\hat{a}_0^2) F(\tau, \lambda_a, \Omega_a) - F(\tau, \lambda_a - 1, \Omega_a)}{\left[1 - \hat{a}_0 \cos(\hat{\omega}_0 \tau)\right]^{1-\lambda}} \right] \right\}, \tag{98}$$

where the integrals $\int_{-\infty}^{\tau} e^{W(\tau' - \tau)} \frac{\sin(\hat{\omega}\tau')d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau')]^{\lambda}}$ and $\int_{-\infty}^{\tau} e^{W(\tau' - \tau)} \frac{\sin(\hat{\omega}\tau')d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau')]^{\lambda}}$ have been evaluated in Appendix B.

10. References

1. J. Oppenlander, Ch. Haussler, and N. Schopohl, Phys. Rev. B63 (2000) 024511.

2. J. W. Swift, S. H. Strogatz, and K. Wiesenfeld, “Averaging of Globally coupled oscillators”. Physica D55 (1992) 239-250

11. Appendix A

The integrals I_1 through I_8 are given by

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\cos(x) - a_0)}{1 - a_0\cos(x)} dx = \frac{\sqrt{1-a_0^2} - 1}{a_0}$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(x)[\cos(x) - c\sin(x)]}{[1 - a_0\cos(x)]^2} dx = \frac{\{2 - b_0^{-1} + 2a_0[1 - a_0(1 - a_0 - a_0^2 + a_0 b_0^{-1})]\}}{2(1-a_0)a_0^2(1+a_0)^2} + \frac{(-1+2a_0^2)}{2(1-a_0)a_0^2(1+a_0)^2\sqrt{1-a_0^2}}$$

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} \frac{[\cos(x) - c\sin(x)]}{[1 - a_0\cos(x)]^2} dx = \frac{a_0}{(1-a_0^2)^{3/2}}$$

$$I_4 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(x)^3}{[1 - a_0\cos(x)]^3} dx = \frac{\pi}{2a_0^3}\{-6 + \frac{[-2a_0 + 4a_0^3 - 2a_0^5 + a_0^2(4 - 5b_0^{-1} + 2(-1 + b_0^{-1}) + a_0^4(-2 + 6b_0^{-1})]}{(1-a_0)^2(1+a_0)^3}$$
$$- \frac{(-2 + 5a_0^2 + 6a_0^4)}{(1-a_0^2)^{5/2}}\}$$

$$I_5 = \frac{1}{2\pi}\int_0^{2\pi} \frac{[\cos(x) - c\sin(x)]^2}{[1 - a_0\cos(x)]^3}dx = \frac{1 + 2a_0^2 + c^2(1 - a_0^2)}{2(1 - a_0^2)^{5/2}}$$

$$I_6 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(x)}{[1 - a_0\cos(x)]^3}dx = \frac{3a_0 b_0^{-1}}{2(1 - a_0)^2(1 + a_0)^3}$$

12. Appendix B

The integral appearing in Eq. 99 may be evaluated by expanding the integrand denominator in a series in powers of $\hat{a}$, which yields

$$\int_{-\infty}^{\tau} e^{W(\tau'-\tau)} \frac{d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau')]^{\lambda}}$$
$$= \int_{-\infty}^{\tau} e^{W(\tau'-\tau)} \left\{1 + \lambda\hat{a}\cos(\hat{\omega}\tau) + \lambda(\lambda+1)\frac{[\hat{a}\cos(\hat{\omega}\tau)]^2}{2!} + \lambda(\lambda+1)(\lambda+2)\frac{[\hat{a}\cos(\hat{\omega}\tau)]}{3!} + \ldots^3\right\}.$$

Each term may now be integrated exactly, and the coefficients of each $\cos(m\hat{w}\tau)$ and $\sin(m\hat{w}\tau)$ may be summed to yield a Fourier series with Fourier coefficients proportional to a hypergeometric function. The final result is given by

$$F(\tau,\lambda,W) = \int_{-\infty}^{\tau} e^{W(\tau'-\tau)} \frac{d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau)]^{\lambda}} = {}_2F_1(\tfrac{\lambda}{2}, \tfrac{\lambda+1}{2}, 1; \hat{a}^2)/W$$

$$+ 2\sum_{n=1}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)n!}\left(\frac{\hat{a}}{2}\right)^n \frac{W\cos(n\hat{\omega}\tau) + n\hat{w}\sin(n\hat{\omega}\tau)}{(n\hat{\omega})^2 + W^2}\, {}_2F_1(\tfrac{n+1+\lambda}{2}, \tfrac{n+\lambda}{2}, n+1; \hat{a}^2).$$

Each term in the summation increases the accuracy of the result by roughly one decimal point.

Integration by parts can be employed to determine the integral

$$\int_{-\infty}^{\tau} e^{W(\tau'-\tau)} \frac{\sin(\hat{\omega}\tau')d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau')]^{\lambda}} = \frac{1}{\hat{\omega}\hat{a}(1-\lambda)}\left\{[1 - \hat{a}\cos(\hat{\omega}\tau)]^{1-\lambda} - WF(\tau,\lambda-1,W)\right\}.$$

The cosine counterpart may be integrated directly to give

$$\int_{-\infty}^{\tau} e^{W(\tau'-\tau)} \frac{cos(\hat{\omega}\tau')d\tau'}{[1 - \hat{a}\cos(\hat{\omega}\tau')]^{\lambda}} = \frac{1}{\hat{a}}\left\{F(\tau,\lambda,W) - F(\tau,\lambda-1,W)\right\}.$$

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